1.0

4.5
5.0
5.6

2.8

2.5

3.2

2.2

3.6

1.1

4.0

2.0

1.8

1.25 1.4 1.6

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

(12)

LEVEL

ADA058319

BRUSHES WITH HIGH CURRENT AND HIGH SLIDING SPEED

AD No.

DC FILE COPY

DDC
RECEIVED
AUG 31 1976

A

NORTHWESTERN UNIVERSITY · QUAECUMQUE · SUNT · VERA ·
ὁ λόγος
πλήρης
χάριτος
καὶ
ἀλη-
θείας
* 1851 *

78 08 80 018

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
THERMOELASTIC EFFECTS IN BRUSHES WITH HIGH CURRENT AND HIGH SLIDING SPEEDS,	Master's thesis June 1977 — May 1978,
	6. PERFORMING ORG. REPORT NUMBER 5341-427

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
Chi-Ping Chen Ralph A. Burton	N00014-75-C-0761 Mod. P00001

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Northwestern University Evanston, IL 60201	

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
Procuring Contacting Officer Office of Naval Research, Dept. of the Navy Arlington, VA 22217	May 1978
	13. NUMBER OF PAGES 26 p.

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report)
	Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Reproduction in whole or in part is permitted for any purpose of the United States Government

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

This report was presented as a M.S. Thesis to the Graduate School, Northwestern University.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

lubrication, thermoelastic, tribology, failure, deformation, electrical contacts, electrical brushes

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Experimental results in the literature indicate that electric brushes may deform, leading to point contact with the slip ring at high current levels. The present work provides theoretical support for these observations and a quantative statement showing how operating variables influence this behavior. The analysis is restricted to a two dimensional model of the brush, with isotropic and constant material properties. The most unexpected result is that, for steady current flow, cooling of the exterior of the brush increases the tendency to deform unfavorably.

DD FORM 1473 EDITION OF 1 NOV 65 IS OBSOLETE
1 JAN 73
S/N 0102-014-6601

TABLE OF CONTENTS

LIST OF SYMBOLS

A	Area of face of brush
E	Young's modulus of brush
e	electrical potential
I	current per unit of thickness of brush
K	thermal conductivity
ℓ	half-width of contact
P	force per unit of thickness holding brush against slip ring
Q	heat flow per unit of brush thickness
q	heat flow per unit of area
r	radial position in transformed Z-plane configuration
R	outer radius of brush
T	temperature
u	displacement
V	sliding speed
W	complex number representing position in W-plane
x	coordinate of position
y	coordinate of position
Z	complex number representing position in Z-plane
α	coefficient of thermal expansion
γ	dimensionless quantity in contact equation
δ	surface displacement
ϵ	$\ell/2$
ζ	position on surface where displacement is given
μ	friction coefficient
ξ	dummy variable
θ	angular coordinate
ρ	electrical resistivity
σ	normal stress
τ	shear stress
ϕ	angular coordinate

W-plane

Fig. 1.   Idealized configuration of brush. Outer surface R is
isopotential, and the contact patch is $-\ell < x < \ell$.

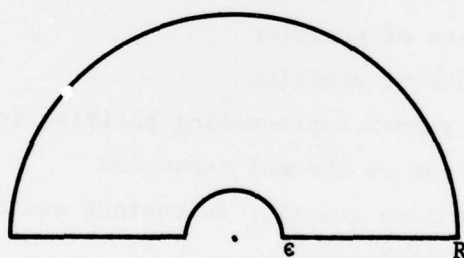Z-plane

Fig. 2.   Axisymmetric configuration, with inner radius ϵ,
and outer radius R.
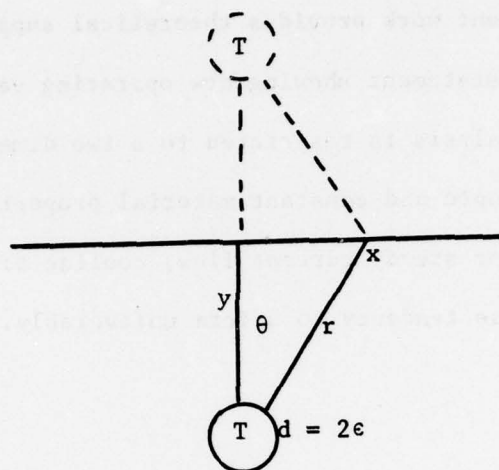
Fig. 3. Illustration for obtaining Green's function for surface displacement produced by small element of elevated temperature T.

I. SUMMARY

Experimental results in the literature indicate that electric brushes may deform, leading to point contact with the slip ring at high current levels. The present work provides theoretical support for these observations and a quantative statement showing how operating variables influence this behavior. The analysis is restricted to a two dimensional model of the brush, with isotropic and constant material properties. The most unexpected result is that, for steady current flow, cooling of the exterior of the brush increases the tendency to deform unfavorably.

II. INTRODUCTION

The electrical brush may, in idealized form, be thought of as a wafer shaped object making contact with the moving surface of a steadily turning cylindrical body, or slip ring. The line of contact would be parallel with the axis of the cylinder, and the sliding would be perpendicular to this line. Even when no current passes through the line of contact, frictional heating may deform the brush, causing contact to transform from a uniformly loaded line to one or more discrete patches. This is the result of the formation of thermal asperities which move slowly across the contact zone. These thermal asperities are often several multiples of the height of the initial roughness or waviness of the brush face, and the peak of each corresponds to a small region or patch of contact with the slip ring, which may be highly stressed as well as hot. Several studies have clarified the nature of the transition to patch-contact in the absence of current flow,[1,2] and Dow has verified the theoretical predictions experimentally.[3] Kilaparti[4] has investigated the role of wear and other factors at work in the limited zone of contact.

Approaching the problem from the electrical side, Marshall has shown that a similar formation of heated contact patches occurs in brushes carrying high levels of current.[5,6] McNab[7] has reviewed the problem of brush wear and has noted ambiguities in the literature as to the combined effects of current, load and speed. Undoubtedly these interactions are complicated by the formation of patch contact.

The present study is intended to provide an analysis which incorporates both frictional heating and current flow, and establishes how these and other factors interact to determine contact patch size, stress level and temperature.

III.   STATEMENT OF THE PROBLEM

A two dimensional brush of unit thickness will be assumed, of
roughly the configuration shown in Fig. 1. It is approximately in
the form of half a circular disk and has a flat side or edge. Near
the center of the edge is the contact patch of width 2ℓ. The outer
circular boundary will be taken as an isopotential line over which
a uniformly distributed current passes into the brush. Current exits
through the contact patch. Assuming the electric field is quasistatic
it will obey LaPlace's equation; and once the potential distribution
is determined, the distribution of electrical heating may be found.

The resultant temperature field can be obtained in two steps. First
a solution is obtained for the condition that electrical heat only flows out
through the contact patch, and that all other boundaries are adiabatic.
Next a solution is found for heat flow into the contact patch and out
through the curved surface of radius R. By superposing these two solutions,
suitably scaled, any combination of heat flows through the patch and outer
surface can be produced.

Once the temperature fields are specified, the thermal displacement
of the boundary can be calculated. In many instances a rounded bulging-
out of the contact patch will be predicted, and through application of
Hertz's theory of contact of cylindrical bodies the conditions can be
found where the contact patch would indeed be pressed smoothly against
the slip ring surface with a contact-free gap to either side.

3

IV. METHOD OF ATTACK

Because the thermal and electrical problems can easily be solved in an axisymmetric configuration (see Fig. 2), it has been found practical to transform these solutions into a very close approximation of the chosen configuration, by conformal transformation. Once the temperature fields are so generated, a simple integral equation can be applied to determine the deflection of the edge of the wafer.

A Green's function for edge deflection can be generated through determination of the deflection distribution produced by a small patch of elevated temperature at an arbitrary position in a semi-infinite plate. This can be found by adaptation of the equations for stress and deflection in a heated axisymmetric body[8] in plane stress. In Fig. 3 is shown the body with the uniform temperature patch of radius ϵ. Disregarding for the moment the broken-line construction also on the figure, we note that for points external to the patch the stresses and deflections are given by:

$$\sigma_r = -\alpha E T \epsilon^2 / 2(x^2 + y^2)$$

$$\sigma_\theta = \alpha E T \epsilon^2 / 2(x^2 + y^2)$$

$$u_r = (1+\nu) \; \alpha T \epsilon^2 / 2\sqrt{x^2 + y^2} \qquad (1)$$

Using the method of images as shown by the broken lines in Fig. 3, we may superpose the fields produced by two temperature sources equidistant from the x-axis, and find that on the x-axis:

$$\sigma_y = 2\sigma_r \; \cos 2\theta$$

$$\sigma_{xy} = 0$$

$$U_y = 0 \qquad (2)$$

For derivation of (1) and (2) see Appendix A.

4

By application of the integral equation derived in Ref. (9) for edge displacements of a plate having an edge load σ_y, we find the normal displacement of the edge, δ, to be:

$$\delta = \frac{2}{\pi E} \int_{-\infty}^{\infty} \sigma_y(\xi)\ell n|\xi-x|d\xi \tag{3}$$

where ξ is a dummy variable. Integrating and letting $\pi\epsilon^2 = dA_i$, one finds that δ_i is the deflection of the plate edge at $x = 0$, caused by a temperature patch $T_i dA_i$, at the position x_i, y_i.

$$\delta_i = \frac{4\alpha T_i}{\pi^2} \left\{ \frac{\pi y_i}{2} + |x_i|\ell n|y_i/x_i| \right\} \frac{dA_i}{x_i^2 + y_i^2}$$

When there is a continuous distribution of temperature in the body the influences may be summed to give, see Appendix B

$$\delta = \frac{4\alpha}{\pi^2} \int_A \left[\frac{\pi y}{2} + |x|\ell n|y/x| \right] \frac{TdA}{(x^2 + y^2)} \tag{4}$$

As it stands this equation gives the displacement at the origin of the coordinate system, but a simple geometric transformation will cause it to apply at $x = \xi$.

$$\delta(\xi) = \frac{4\alpha}{\pi^2} \int_A \left[\frac{\pi y}{2} + |x-\xi|\ell n\left|\frac{y}{x-\xi}\right| \right] \frac{TdA}{y^2+(x-\xi)^2} \tag{5}$$

It is of practical interest that the second term in the brackets has only a small influence on relative displacements in the contact patch.

V. TEMPERATURE SOLUTIONS IN THE AXISYMMETRIC PLATE

For the case of simple conduction through the cylindrical surface of radius ϵ, see Fig. 2, LaPlace's equation for temperature will be satisfied by

$$T_c - T(r) = (Q_b/K\pi)\ell n(r/\epsilon) \tag{6}$$

where Q_b is the heat flow moving radially outward, T_c is the temperature of the inner cylindrical surface and $T(r)$ is the temperature at arbitrary r interior to R.

Turning now to the question of electric current flow, one finds by analogy

$$e_c - e(r) = \frac{\rho I}{\pi}\ell n(r/\epsilon) \tag{7}$$

Noting that electric heat generation will be given by

$$q = \frac{1}{\rho}\left(\frac{de}{dr}\right)^2 \tag{8}$$

it follows that the total electrical heating between r and R is given by

$$Q(r) = \frac{I^2\rho}{\pi}\ell n\left(\frac{R}{r}\right) \tag{9}$$

irrespective of the direction of current flow. The maximum magnitude of Q would be

$$Q_{e\ell} = \frac{I^2\rho}{\pi}\ell n(R/\epsilon) \tag{10}$$

and would be realized as heat flow through the inner surface, if all heat flow were blocked at the boundary, R. Under the same boundary condition, we may write at any r,

$$-K\frac{dT}{dr} = Q(r) = \frac{I^2\rho}{\pi}\ell n\frac{R}{r}$$

6

and it follows that

$$T(r) - T(R) = \frac{Q_e}{2K\pi} \frac{[\ell n(r/R)]^2}{\ell n(R/\epsilon)} \tag{11}$$

This may be rewritten as

$$T(r) - T(R) = \frac{Q_e}{2K\pi} \left\{ \ell n(R/\epsilon) + 2\ell n(\epsilon/r) + \frac{[\ell n(\epsilon/r)]^2}{\ell n(R/\epsilon)} \right\} \tag{12}$$

VI. CALCULATION OF SURFACE DEFLECTION

As stated above, solutions in the W-plane were to be obtained from the axisymmetric Z-plane solutions by conformal transformation. To this e... the Zhukovsky transform may be applied.

$$W = Z + 1/Z \tag{13}$$

This may be rewritten as

$$Z = [W + \sqrt{W^2 - 4}]/2 \tag{14}$$

Hence for any W, expressed as a complex number $x + iy$, there will be a Z, which can be expressed as $re^{i\varphi}$. Knowing r, one may use Eq. (6) or (12) as appropriate, to obtain the temperature at the point and at the corresponding W-point. The use of the transform on the electrically heated field is permissible because the quantity

$$q = \frac{1}{\rho} \left[\left(\frac{\partial e}{\partial x}\right)^2 + \left(\frac{\partial e}{\partial y}\right)^2 \right] \tag{15}$$

transforms such that

$$q(Z) = q(W) \left|\frac{dW}{dZ}\right|^2 \tag{16}$$

See Ref. (10). Once values of T are known at points in the W-plane it is possible to evaluate the integral of Eq. (5) at various values of ξ. Closed form solution was possible only for the case of simple heat transfer (see Eq. (6)). Consequently, the space was subdivided into a net of points and the influences of each were summed. The simple heat flow solution provided a check on the accuracy of the summation procedure. Results are reported here as the difference in displacement between contact center and edge or

$$\hat{\delta} = \delta(0) - \delta(\ell) \tag{17}$$

8

For the case of simple heat flow radially outward, the exact solution is:

$$\hat{\delta}_{th} = \frac{\ell \alpha Q_b}{\pi K} (0.5708) \tag{18}$$

By numerical integration (for $R/\epsilon = 1000$)

$$\hat{\delta}_{th} = \frac{\ell \alpha Q_b}{\pi K} (0.5715) \tag{19}$$

See Appendix C.

The close agreement gives confidence in the numerical procedure, so it may be applied to the case of electrical heating with all of the heat passing outward through the contact patch, drawing upon Eq. (12) for the temperatures; and noting that any component of the temperature distribution that is constant throughout the field will not contribute to deformation of the boundary, therefore it will not contribute to $\hat{\delta}$. The result of this numerical integration is

$$\hat{\delta}_e = \frac{\ell \alpha Q_e}{\pi K} \left[-0.5708 + \frac{0.7636}{\ell n(R/\epsilon)} \right] \tag{20}$$

This result is interesting in two respects. First it represents an indentation rather than a protrusion of the surface, and second it is dependent on the size of the brush, R. In the first we note simply that for heat to be removed through the contact patch the lowest temperature must be at the contact surface, hence a deficit of temperature there and the resultant indentation.
The second effect is the result of the fact that the more material through which current is passed the more heat generated.

The combined displacement for electrical heating and brush cooling may be obtained by superposition of

$$\hat{\delta} = \hat{\delta}_e + \hat{\delta}_{th}$$

or

$$\hat{\delta} = \frac{\ell \alpha Q_e}{\pi K} \left[0.5708(\frac{Q_b}{Q_e} - 1) + \frac{0.7636}{\ell n(R/\epsilon)} \right] \tag{21}$$

Since Q_b represents heat passing through the outer surface of the brush it may be thought of as a measure of cooling. If the brush were insulated Q_b would be zero. With some forced cooling Q_b/Q_e might be any value up to or exceeding unity.

9

VII. COMBINED EFFECTS OF FRICTIONAL AND ELECTRICAL HEATING

Frictional heating of the contact patch is given by

$$Q_f = \mu V P$$

where V is sliding speed, P is contact load and μ is friction coefficient. Recalling that

$$Q_e = (I^2 \rho / \pi) \ln(2R/\ell) \tag{22}$$

one may write

$$Q_{total} = Q_e + Q_f \tag{23}$$

We now note that with isolated slip ring and cooled brush Q_b could have as its upper limit Q_{total}. Hence the maximum value of $\hat{\delta}$, given by Eq. (22) would be

$$\hat{\delta}_{max} = \frac{\ell\alpha}{\pi K} \left[0.5708(Q_{tot} - Q_e) + \frac{0.7636}{\ln(R/\epsilon)} Q_e \right] \tag{24}$$

For convenience call the quantity in brackets γ, (in Eq. (24)) making

$$\hat{\delta} = (\alpha \ell / \pi K)(\gamma) \tag{25}$$

Let us now draw upon the well known relationship for Hertzian contact

$$\hat{\delta} = P/1.72E \tag{26}$$

where $\hat{\delta}$ is the distance the center of the contact is indented relative to the edges of the contact. It follows that if the thermal bump is pressed flat, to assure full contact over the region 2ℓ,

$$\frac{P}{1.72E} = \left[\frac{\alpha \ell}{K\pi} \right] \left[\gamma \right] \tag{27}$$

10

Here it is assumed that the deflection of the slip ring is negligible relative to that of the carbon, therefore only the Young's modulus E of the carbon brush need be considered.

In Eq. (27) it is seen that ℓ increases with brush load P. Increasing of cooling of the brush causes γ to increase and causes ℓ to be smaller for a given load, an adverse condition. Although the derivation is not valid for large ℓ relative to brush size R, nevertheless the condition $\ell = R$ provides a criterion as to when thermal effects are beginning to be important, and for $\ell \ll R$ the equation should serve to predict contact patch size.

To obtain contact temperature one need only to refer to Eq. (5,12) and superpose the magnitudes of $T(r = \epsilon)$, thus giving the elevation of contact temperature above brush temperature $T(R)$. To obtain contact stress note that its mean value would be $P/2\ell$.

REFERENCES

1. T. A. Dow and R. A. Burton, "Thermoelastic Instability of Sliding Contact in the Absence of Wear," Wear, 19, pp. 315-328, (1972).

2. T. A. Dow and T. A. Burton, "The Role of Wear in the Initiation of Thermoelastic Instabilities in Rubbing Contact," ASME Trans., Ser. F, 95, pp. 71-75, (1973).

3. T. A. Dow and R. D. Stockwell, "Experimental Verification of Thermoelastic Instabilities in Sliding Contact," ASME Trans., Ser. F, 261, pp. 359-364, (1977).

4. S. R. Kilaparti and R. A. Burton, "The Thermoelastic Patch Contact Problem for Large Peclet Number," ASME Trans., Ser. F, 100, pp. 65-69, (1978).

5. R. A. Marshall, "The Mechanism of Current Transfer in High Current Sliding Contacts," Wear, 37, pp. 233-240, (1976).

6. R. A. Marshall, "Design of Brush Gear for High Current Pulses and High Rubbing Velocities," IEEE Trans. Power Apparat. Syst., PAS 85, pp. 1177-1188, (1966).

7. I. R. McNab, "Pulsed High Power Brush Research," Electrical Contacts 1977, R. Armington, Ed., Illinois Institute of Technology, Chicago, pp. 107-114, (1977).

8. S. Timoshenko and J. N. Goodier, Theory of Elasticity, Third Ed. New York: McGraw-Hill, 1970, pp. 445-446.

9. Ibid, pp. 107-109.

10. R. Churchill, Introduction to Complex Variables and Applications. New York: McGraw-Hill.

11. J. Dundurs, Mechanics Research Communications, Pergamon Press, 1, 1974, pp. 121-124.

APPENDIX A

DERIVATION OF STRESSES AND DEFLECTIONS FOR POINTS EXTERNAL TO THERMAL PATCH

From reference 1, one can find the expressions for a thin circular disk with the edge r = b free from stress:

$$\sigma_r = \alpha E \left(\frac{1}{b} \int_0^b Trdr - \frac{1}{r^2} \int_0^r Trdr \right)$$

$$\sigma_\theta = \alpha E \left(\frac{1}{b^2} \int_0^b Trdr + \frac{1}{r^2} \int_0^r Trdr - T \right) \qquad (A-1)$$

$$u = (1+\nu)\alpha \frac{1}{r} \int_0^r Trdr + (1-\nu)\alpha \frac{r}{b^2} \int_0^b Trdr$$

For a body with a uniform temperature patch of radius ϵ, the integral in the brackets is

$$\int_0^r Trdr = \int_0^\epsilon Trdr + \int_\epsilon^b Trdr = \frac{T\epsilon^2}{2}$$

$$\int_0^r Trdr = \int_0^\epsilon Trdr + \int_\epsilon^r Trdr = \frac{T\epsilon^2}{2} \qquad (A-2)$$

Where the second integral is zero when the temperature is zero outside of patch. And note that $\pi\epsilon^2 = dA$.

For $b \gg \epsilon$.

$$\sigma_r = \alpha E \left(\frac{1}{b} - \frac{1}{r^2} \right) \frac{TdA}{2\pi} = - \frac{\alpha E TdA}{2\pi(x^2+y^2)}$$

$$\sigma_\theta = \alpha E \left(\frac{1}{b^2} \frac{TdA}{2\pi} + \frac{1}{r^2} \frac{TdA}{2\pi} - T \right) = \frac{\alpha E TdA}{2\pi(x^2+y^2)}$$

$$u = (1+\nu)\alpha TdA/2\pi \sqrt{x^2+y^2}$$

13

Using the method of images as shown by the broken lines in Fig. 3, we may superpose the fields produced by two temperature sources equidistant from the x-axis, and note that by Mohr's circle one can find that

$$\sigma_y = 2\sigma r \cos 2\theta \qquad (A-3)$$

$$\sigma_{xy} = 0$$

$$u_y = 0$$

APPENDIX B

DEFLECTION OF THE PLATE EDGE

From (3) one finds that the deflection of the plate edge at $x = 0$ is

$$\delta_i = \frac{2}{\pi E} \int_{-\infty}^{\infty} \sigma_{y_i}(\xi) \ln|\xi - x_i| \, d\xi \tag{B-1}$$

from (2)

$$\sigma_{y_i}(\xi) = 2\sigma_r \cos 2\theta$$

$$= 2\sigma_r(1 - 2\cos^2\theta)$$

$$= 2\left[-\frac{\alpha E T_i dA_i}{2\pi(\xi^2 + y_i^2)}\right]\left[1 - 2\frac{y_i^2}{\xi^2 + y_i^2}\right]$$

$$= -\frac{\alpha E T_i dA_i}{\pi}\left[\frac{1}{\xi^2 + y_i^2} - \frac{2y_i^2}{(\xi^2 + y_i^2)^2}\right] \tag{B-2}$$

Substitute (B-2) into (B-1) and note that deflection is symmetric to y-axis.

$$\delta_i = -\frac{4\alpha T_i dA_i}{\pi^2}\left[\int_0^{\infty} \frac{\ln|\xi - x_i|}{\xi^2 + y_i^2} \, d\xi - 2y_i^2 \int_0^{\infty} \frac{\ln|\xi - x_i|}{(\xi^2 + y_i^2)^2} \, d\xi\right] \tag{B-3}$$

$$\int_0^{\infty} \frac{\ln|\xi - x_i|}{\xi^2 + y_i^2} \, d\xi - 2y_i^2 \int_0^{\infty} \frac{\ln|\xi - x_i|}{(\xi^2 + y_i^2)^2} \, d\xi$$

$$= \int_0^{x_i} \frac{\ln(x_i - \xi)}{y_i^2 + \xi^2} \, d\xi + \int_{x_i}^{\infty} \frac{\ln(\xi - x_i)}{y_i^2 + \xi^2} \, d\xi \tag{B-4}$$

$$- 2y_i^2 \int_0^{x_i} \frac{\ln(x_i - \xi)}{(\xi^2 + y_i^2)^2} \, d\xi - 2y_i^2 \int_{x_i}^{\infty} \frac{\ln(\xi - x_i)}{(\xi^2 + y_i^2)^2} \, d\xi$$

One can integrate each term in (B-4) by parts as following:

$$\int_0^{x_i} \frac{\ln(x_i - \xi)}{y_i^2 + \xi^2} \, d\xi = \frac{\ln(x_i - \xi)}{y_i} \tan^{-1}\frac{\xi}{y_i}\Big|_0^{x_i} + \frac{1}{y_i} \int_0^{x_i} \frac{\tan^{-1}\frac{\xi}{y_i}}{x_i - \xi} \, d\xi$$

$$\int_{x_i}^{\infty} \frac{\ln(\xi - x_i)}{y_i^2 + \xi^2} \, d\xi = \frac{\ln(\xi - x_i)}{y_i} \tan^{-1} \frac{\xi}{y_i} \Big|_{x_i}^{\infty} - \frac{1}{y_i} \int_{x_i}^{\infty} \frac{\tan^{-1} \frac{\xi}{y_i}}{\xi - x_i} \, d\xi$$

$$\int_{o}^{x_i} \frac{\ln(x_i - \xi)}{(y_i^2 + \xi^2)^2} \, d\xi = \frac{\ln(x_i - \xi)}{2y_i^2} \left[\frac{\xi}{y_i^2 + \xi^2} + \frac{1}{y_i} \tan^{-1} \frac{\xi}{y_i} \right]_{o}^{x_i}$$

$$+ \int_{o}^{x_i} \frac{1}{2y_i^2} \left[\frac{\xi}{y_i^2 + \xi^2} + \frac{1}{y_i} \tan^{-1} \frac{\xi}{y_i} \right] \frac{d\xi}{x_i - \xi}$$

$$\int_{x_i}^{\infty} \frac{\ln(\xi - x_i)}{(y_i^2 + \xi^2)^2} \, d\xi = \frac{\ln(\ell - x_i)}{2y_i^2} \left[\frac{\xi}{y_i^2 + \xi^2} + \frac{1}{y_i} \tan^{-1} \frac{\xi}{y_i} \right]_{x_i}^{\infty}$$

$$- \int_{x_i}^{\infty} \frac{1}{2y_i^2} \left[\frac{\xi}{y_i^2 + \xi^2} + \frac{1}{y_i} \tan^{-1} \frac{\xi}{y_i} \right] \frac{d\xi}{\xi - x_i}$$

After substitution one can arrive.

$$\delta_i = \frac{4\alpha T_i dA_i}{\pi^2} \left[\int_{o}^{\infty} \frac{\xi d\xi}{(y_i^2 + \xi^2)(\xi - x_i)} \right]$$

$$= \frac{4\alpha T_i dA_i}{\pi^2} \left[\frac{\pi y_i}{2} + |x_i| \ln \left| \frac{y_i}{x_i} \right| \right] \frac{1}{x_i^2 + y_i^2}$$

When there is a continuous distribution of temperature in the body, the influence may be integrated over the whole body to give

$$\delta = \int_{A} \delta_i dA = \frac{4\alpha}{\pi^2} \int_{A} \left[\frac{\pi y}{2} + |x| \ln \left| \frac{y}{x} \right| \right] \frac{T dA}{(x^2 + y^2)} \tag{B-5}$$

APPENDIX C

EXACT SOLUTION

For the case of simple conduction the heat flow through the contact line segment in W-plane is

$$q_W = q_z \left| \frac{dZ}{dW} \right| \tag{C-1}$$

where q_W and q_z are heat flow per unit length in W-plane and Z-plane. If we let ϵ equal to unity in Z-plane, then

$$q_z = \frac{Qb}{\pi\epsilon} = \frac{Qb}{\pi} \tag{C-2}$$

From (14) one can get

$$\left| \frac{dZ}{dW} \right| = \frac{1}{2\sqrt{1-\left(\frac{W}{2}\right)^2}} \tag{C-3}$$

the semi unit circle transforms into a line segment in W-plane. Also, one can find that[11] the curvature in W-plane is

$$\frac{d^2\delta'}{dx'^2} = \frac{-\alpha q_w}{K} \tag{C-4}$$

here for q_w heat flow moving outward, the curvature is negative. let $x' = \frac{x}{2}$ $|x'| \le 1$ and

$$\frac{d^2\delta'}{dx^2} = \frac{-4q_w\alpha}{K} = \frac{-4\alpha q_z}{K}\left| \frac{dZ}{dW} \right|$$

$$= \frac{-2\alpha Q_b}{K\pi \sqrt{1-x'^2}} \tag{C-5}$$

integrate (C-5) twice and note dimensional deflection $\delta = \delta' \frac{\ell}{2}$ one can get

$$\delta(x') = -\frac{\ell\alpha Q_b}{\pi K}\left[x' \sin^{-1} x' + \sqrt{1-(x')^2} \right] + C$$

here c is a constance.

17

$$\hat{\delta} = \delta(0) - \delta(1) = \frac{\ell \alpha Q_b}{\pi K} \left(\frac{\pi}{2} - 1\right)$$

$$= \frac{\ell \alpha Q_b}{\pi K} (0.5708) \tag{C-6}$$

APPENDIX D

NUMERICAL SOLUTION

From (5) and (6) we have

$$\delta'(\xi) = \frac{-4\alpha}{\pi^2} \int_A \left[\frac{\pi y}{2} + |x-\xi| \ln \left| \frac{y}{x-\xi} \right| \right] \frac{Q_b}{K\pi} \left(\frac{R}{r} \right) \frac{dA}{y^2 + (x-\xi)^2}$$

note that

$$dA_W = dA_z \left| \frac{dW}{dz} \right|^2$$

$$= \frac{r^4 - 2r^2 \cos 2\phi + 1}{r^3} \, dr d\phi \tag{D-1}$$

$$x = \left(r + \frac{1}{r} \right) \cos \phi$$

$$y = \left(r - \frac{1}{r} \right) \sin \phi$$

one can integrate from r=1 to R ϕ=0 to π numerically, and note that

dimensional $\delta = \frac{\ell}{2} \delta'$. The final result is

$$\delta = \delta(0) - \delta(2) = \frac{\ell \alpha Q_b}{\pi K} (0.5715) \tag{D-2}$$

19

COMPUTER LISTING

```
        PROGRAM DEF (OUTPUT)
C       NUMERICAL INTEGRATION OF DEFLECTION
C       A = ANGLE FROM 0 TO 3.1416
C       RADIUS INCREMENT = GA
C       ANGULAR INCREMENT = G * 1.5708
        R = 1000.0
        G = 0.002
        P = 0.0
1       D = 0.0
        GA = 0.0015
        SR = 1.0 + GA / 2.0
2       A = G * 1.5708 / 2.0
3       A2 = A + A
        S1 = SIN (A)
        C1 = COS (A)
        C2 = COS (A2)
        T=(SR**4.0-2.0*SR*SR*(C2+1.0) / SR**3.0
        U1 = (SR + 1.0/SR ) * C1
        V = (SR -1.0 / SR ) * S1
        U2 = U1 - P
        U = ABS (U2)
        S = U / V
        Q=(3.1416*V/2-U*ALOG(S)) / (U*U + V*V)
        F1 = R / SR
        F = ALOG (F1)
        DN = F * T * Q
        DN = DN * G * 1.5708 * GA
        D = D + DN
        A = A + G * 1.5708
        IF ( A .LE. 3.1416 ) GO TO 3
        IF ( P .NE. 0.0 ) GO TO 4
        PRINT 300 , SR, GA
4       SR = SR + GA / 2.0
        GA = 1.2 * GA
        SR = SR + GA / 2.0
        IF (SR .LE. R ) GO TO 2
        IF (P .NE. 0.0 ) GO TO 20
        DC = D
20      X = ( D - DC ) * 2.0 / 3.141593 ** 2.0
        PRINT 200 , R, P , X , D
        P = P + 2.0
        IF ( P .LE. 2.0 ) GO TO 1
200     FORMAT (2(5X,F4.1), 2(5X,F20.6))
300     FORMAT (2(10X,F10.4))
        STOP
        END
```

20